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SOME FIXED POINT THEOREMS IN DISLOCATED QUASI ULTRAMETRIC SPACES USING DIFFERENT CONTRACTIONS

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Abstract. This paper generalizes fixed point results in non-traditional metric spaces by introducing dislocated quasi-ultrametric spaces (dq-ultrametric spaces). It establishes fixed point existence and uniqueness using specific contractions, explores implications for integral equations, and expands upon prior findings. Key results are illustrated with examples, broadening the ultrametric framework's applicability.

Keywords: contraction mapping, Cauchy sequence, dq-metric space, fixed points, ultrametric space.

AMS Subject Classification: 47H10, 54H25.

1. Introduction

The fixed point theory is an essential part of nonlinear analysis, and it has been used to prove that the solutions of different mathematical models exist and to prove the uniqueness of the solution. Several authors have made significant contributions to this theory through their writings (see [2, 10, 17, 22]). An essential theorem of fixed point theory is the Banach contraction principle, which says that there is only one fixed point for every contraction in a complete metric space. The concept of metric space has been extended in various ways. Some notable and impact full generalizations of metric spaces include b-metric space, cone metric space, cone b-metric space, dislocated metric space, quasi-metric space, dq-metric space, generalized quasi-metric space, and so on. Mainly, topological approaches are used to obtain fixed-point semantics in logic programs. Such considerations inspired dislocated metric spaces. The dislocated topology implementations have been examined in the context of logic programming semantics (see [14]). They established the concept of dislocated metric space and modified the Banach contraction theorem in such spaces to achieve a unique supported model for these applications. Furthermore, The concept of completely dq-metric spaces was introduced to generalize previous results [14, 24, 39]. Fixed point theorems in dq-metric spaces were developed in [1], while some fixed point results under continuously contractive conditions with a rational type expression were established in [15]. Further generalization of these results was presented through a fixed point theorem in [20]. Additional results in dislocated and dq metric spaces were provided in [41]. Fixed point

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theorems for continuous self-mappings in dq-metric spaces were investigated in [37], and new fixed point results were constructed in [28].

Aage and Salunke developed fixed point theorems in dq-metric spaces [1]. Isufati [15] also found some fixed point results in a dq-metric space for conditions that are continuously contractive and have a rational type expression. Kohli et al. [20] investigated a fixed point theorem that generalized Isufati's result. Zoto presented some new results in dislocated and dq-metric spaces in [41]. Shrivastava et al. [37] investigated a fixed point theorem in dq-metric spaces for a continuous self-mapping. Patel and Patel [28] built some new fixed points that result in a dq-metric space.

In mathematics, we can obtain a dislocated quasi-metric space by subtracting one and a half restrictions from the three restrictions of a metric space. A complete dislocated quasi-metric space is a more generalized version of a complete quasi-partial metric space and an orthogonal-complete space, as stated in references [3, 13, 19]. Additionally, a dislocated quasi-metric space also encompasses the concepts of dislocated metric and partial metric. References [8, 36, 40] contain the fixed point results established by several researchers in dislocated quasi-metric space.

The field of non-Archimedean functional analysis was initiated by Monna, who published a series of papers in 1943. A significant development occurred in 1978 with the publication of book [38], which remains the most comprehensive resource on non-Archimedean Banach spaces in the literature. For a metric space (X, d) to be ultrametric (non-Archimedean), it must satisfy the stronger triangle inequality, i.e., $d(x, y) \leq \max\{d(x, z), d(y, z)\},\ for\ \ all\ x, y, z \in X$. Note that any ultrametric space is also a metric space, but the converse is not necessarily true. The concept of ultrametric spaces has practical applications, such as in taxonomy and phylogenetic tree construction.The notion of ultrametric spaces was introduced in [38], while a fixed point theorem for a class of generalized contractive mappings in an ultrametric space was proved in [12]. Furthermore, two coincidence point theorems for three and four self-maps in a spherically complete ultrametric space were introduced in [33]. Fixed point results on ultrametric spaces have also been established in and related studies on fixed point theorems can be found in [23, 29]. In 2016, fixed-point results for dislocated quasi ultrametric spaces were reported in [6]. These spaces have unique characteristics that differ from those of other metric spaces or general modular spaces. The spaces exhibiting ultrametric inequality possess unique geometric and topological features that can result in different outcomes compared to standard metric spaces. Theorems established for ultrametric spaces are tailored to their exceptional properties. In 2017, fixed point theorems in partially ordered ultrametric and non-Archimedean normed spaces were established in [21], focusing on single-valued and strongly contracted mappings. Additionally, mappings in ultrametric spaces involving contractions and set-valued contractions were analyzed in [32]. Recent studies in [30] employed p-adic distance to derive novel fixed-point theorems on partially ordered ultrametric spaces. Furthermore, common fixed point results in dislocated quasi-ultrametric spaces were obtained in [7] by utilizing different contractions and demonstrating their applications. Moreover, new coincidence point theorems were introduced in partially ordered ultrametric spaces using extended rational contractions in [31]. These advancements underline the growing importance of ultrametric spaces in modern mathematical analysis. As the field continues to evolve, further exploration of these concepts could yield additional insights into both fixed point theory and its applications across various scientific domains (see [4, 5, 11, 18, 25-27, 34, 35]).

An ultrametric spaces are a specific type of metric space that has a unique property known as ultrametric inequality. This property distinguishes them from regular metric spaces. In an ultrametric space, the distance between two points is always less than or equal to the maximum of their distances from a third point. This is different from the traditional triangle inequality in regular metric spaces. The study of fixed points in ultrametric spaces can lead to many intriguing and unexpected results. The Banach Fixed Point Theorem for ultrametric spaces is one of the most well-known theorems in this area. It states that if a contraction mapping is applied to a complete ultrametric space, it will have a single, unique fixed point. A contraction mapping is a function that decreases the distances between points, while completeness ensures that the space contains all its limit points.

Definition 1.1. [9] *Consider a fixed prime number p. Also, let $c \in \mathbb{R}$, where $0 < c < 1$ and c will be fixed throughout the discussion. If $\varkappa$ is any rational number other than zero, we can write $\varkappa$ in the form*

$$\varkappa = p^{\delta}\frac{u}{v},$$

where $\delta \in \mathbb{Z}$, and $u, v \in \mathbb{Z}$, and $p \nmid a$,$p \nmid b$, and clearly, $\sum \delta$ may be positive, negative or zero, we now define

$$|\varkappa|_p = c^{\delta} \text{ and } |0|_p = 0.$$

It follows immediately from the definition that, $|\varkappa|_p \geq 0$ if and only if $\varkappa = 0$.

Example 1.1. [9] Take $\varkappa = \frac{19}{216}$. Suppose if we want to find its 2-adic absolute value (where p=2), first, we write $\varkappa$ in the following form

$$\varkappa = \frac{19}{8 \times 27} = 2^{-3} \times \frac{19}{27},$$

which implies that $|\varkappa|_2 = 2^3 = 8$. Then, what about its 19-adic absolute value? It will simply be $|\varkappa|_{19} = \frac{1}{19}$ because

$$\varkappa = 19 \times \frac{1}{216} \text{ thus } |\varkappa|_{19} = \frac{1}{19}.$$

Also, it is trivial that the p-adic absolute value of a rational number when p divides neither the numerator nor the denominator is 1, since $p^0 = 1$.

1.1. Motivation of this study.

Definition 1.2. [16] *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. Then T is called a Kannan mapping if*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + d(y, Ty),$$

for all $x, y \in X$ and $\alpha \in [0, 1)$.

Theorem 1.1. [16] *Kannan established a unique fixed point theorem for a mapping that satisfies the above condition in metric spaces.*

Besides that, we came up with the following fixed point theorems by using a generalized contraction and a Kannan-type contraction in the context of quasi-metric spaces that are dislocated [1].

Theorem 1.2. [1] *Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the following condition:*

$$d(Tx, Ty) \leq a \cdot d(x, Tx) + d(y, Ty),$$

for all$x, y \in X$, where $a \geq 0$ with $a < 1$. Then T has a unique fixed point.

1.2. **Structure of this study.**

This research article is divided into four sections. In Section 1, we present an introduction that provides the foundational background necessary for proving the primary results of our investigation. Section 2 focuses on the main results. In Section 3, we demonstrate the applications of our results to integral equations. Finally, in Section 4, we conclude our work with a summary of findings and potential directions for future research.

The aim of this paper is to investigate the existence and uniqueness of fixed points in dq-ultrametric spaces. Utilizing various contraction principles, we derive several corollaries for fixed points of self-mappings that satisfy more generalized contractive conditions in such spaces. To support our findings, we provide illustrative examples. Additionally, we explore the application of integral equations in the context of fixed points.

2. Main results

Definition 2.1. *Consider a non-empty set Y and a function $d : Y \times Y \to [0, +\infty)$ that satisfies the following conditions:*

(d_1) $d(\varkappa, \varkappa) = 0$;

(d_2) $d(\varkappa, \gamma) = d(\gamma, \varkappa) = 0$, *only if* $\varkappa = \gamma$;

(d_3) $d(\varkappa, \gamma) = d(\gamma, \varkappa)$, *for all* $\varkappa, \gamma \in Y$;

(d_4) $d(\varkappa, \mathsf{c}) \leq \max\big(d(\varkappa, \gamma), d(\gamma, \mathsf{c})\big)$ *for all* $\varkappa, \gamma, \mathsf{c} \in Y$.

If d satisfies all the four conditions, it is called an ultrametric on Y. If d satisfies conditions (d_2) to (d_4), it is called a dislocated ultrametric space on Y. Finally, if d satisfies conditions (d_2) and (d_4) only, then it is called a dq*-ultrametric on Y.*

It is important to note that every metric on Y is a dislocated metric on Y. However, the converse is not necessarily true, as demonstrated by the following example:

Example 2.1. Let $Y = [0, 1]$, we define the function $d_u : Y \times Y \to \mathbb{R}^+$ as

$$d_u(\varkappa, \gamma) = \max\big\{|\varkappa - \gamma|, |\varkappa|\big\}.$$

Then d_u is a dislocated ultrametric but d_u is not a metric.

Example 2.2. Let $Y = \mathbb{R}$ and let $d_u(\varkappa, \gamma) = |\varkappa - \gamma| + \frac{|\varkappa|}{n} + \frac{|\gamma|}{m}$, where $m, n \in \mathbb{N}$ with $n \neq m$. Then (Y, d_u) is a dislocated quasi ultrametric space.

Let $\varkappa, \gamma, \mathsf{c} \in Y$, suppose that $d_u(\varkappa, \gamma) = 0$. Then $|\varkappa - \gamma| + \frac{|\varkappa|}{n} + \frac{|\gamma|}{m} = 0$.

It implies that $|\varkappa - \gamma| = 0$, and so $\varkappa = \gamma$. Next, consider

$$\begin{aligned} d_u(\varkappa, \gamma) &= |\varkappa - \gamma| + \frac{|\varkappa|}{n} + \frac{|\gamma|}{m} \\ &\leq |\varkappa - \mathsf{c} + \mathsf{c} - \gamma| + \frac{|\varkappa|}{n} + \frac{|\gamma|}{m} \\ &\leq \max\big[|\varkappa - \mathsf{c}|, |\mathsf{c} - \gamma|\big] + \frac{|\varkappa|}{n} + \frac{|\gamma|}{m} + \frac{|\mathsf{c}|}{m} + \frac{|\mathsf{c}|}{n}. \end{aligned}$$

Thus

$$d_u(\varkappa, \mathsf{c}) \leq \max\big(d_u(\varkappa, \gamma), d_u(\gamma, \mathsf{c})\big).$$

Definition 2.2. *A sequence $\{\varkappa_n\}$ in a* dq*-ultrametric space is called a Cauchy sequence if for $\epsilon > 0$ there exists a positive integer N such that for $n \geq N$, we have $d_u(\varkappa_n, \varkappa_{n+1}) \leq \epsilon$.*

Definition 2.3. *A sequence $\{\varkappa_n\}$ is called* dq*-convergent in Y if for $n \geq N$, we have $d_u(\varkappa_n, \varkappa) \leq \epsilon$, where $\varkappa$ is called the* dq*-limit of the sequence $\{\varkappa_n\}$.*

Definition 2.4. *A* dq*- ultrametric space (Y, d_u) is said to be complete if every Cauchy sequence in Y converges to a point Y.*

2.1. **Fixed point theorems related to existence and uniqueness in dislocated quasi-ultrametric space.**

Theorem 2.1. *Let (Y, d_u) be a complete* dq*-ultrametric space, and let $\top : Y \to Y$ be a continuous self mapping satisfying the following contraction*

$$d_u(\top\varkappa, \top\gamma) \leq \delta d_u(\varkappa, \gamma) + \beta d_u(\gamma, \top\varkappa) + \gamma d_u(\varkappa, \top\gamma) \tag{1}$$

where $\delta, \beta, \gamma \in [0, 1)$ with $\delta + 2\beta + \gamma < 1$. Then $\top$ has a unique fixed point.

Proof. Let $\varkappa_0$ be chosen arbitrarily. Then we define a sequence $\{\varkappa_n\}$ by the rule $\varkappa_0\varkappa_1 = \top\varkappa_0$, $\varkappa_2 = \top\varkappa_1, \cdots \varkappa_{n+1} = \top\varkappa_n$, for all $\mathfrak{n} \in \mathbb{N}$.

Now we show that $\varkappa_n$ is a Cauchy sequence in Y. Suppose

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &= d_u(\top\varkappa_{n-1}, \top\varkappa_n) \\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_n, \top\varkappa_{n-1}) + \gamma d_u(\varkappa_{n-1}, \top\varkappa_n) \\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_n, \varkappa_n) + \gamma d_u(\varkappa_{n-1}, \varkappa_{n+1}) \\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_n, \varkappa_n) + \gamma \max\Big\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\Big\}.
\end{aligned}$$

Case 1: If $\max\Big\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\Big\} = d_u(\varkappa_{n-1}, \varkappa_n)$ then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_n, \varkappa_{n+1}) + \gamma d_u(\varkappa_{n-1}, \varkappa_n) \\
&\leq \left(\frac{\delta + \beta + \gamma}{1 - \beta}\right) d_u(\varkappa_{n-1}, \varkappa_n) \\
&\leq \mathsf{k} \;\; d_u(\varkappa_{n-1}, \varkappa_n),
\end{aligned}$$

where $\mathsf{k} = \frac{\delta+\beta+\gamma}{1-\beta} < 1$. Thus we have

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{k}^n \;\; d_u(\varkappa_0, \varkappa_1) \qquad \text{for all } \mathfrak{n} \in \mathbb{N}.$$

Case 2: If $\max\Big\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\Big\} = d_u(\varkappa_n, \varkappa_{n+1})$, then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_n, \varkappa_{n+1}) + \gamma d_u(x_n, x_{n+1} + b d_u(\varkappa_n, \varkappa_{n+1})(1 - \gamma) d_u(\varkappa_n, \varkappa_{n+1} \\
&\leq \left(\frac{\delta + \beta}{1 - \beta - \gamma}\right) d_u(\varkappa_{n-1}, \varkappa_n) \leq \mathsf{h} \;\; d_u(\varkappa_{n-1}, \varkappa_n),
\end{aligned}$$

where $\mathsf{h} = \frac{\delta+\beta}{1-\beta-\gamma} < 1$. So we have

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{h}^n \;\; d_u(\varkappa_0, \varkappa_1) \qquad \text{for all } \mathfrak{n} \in \mathbb{N}.$$

Since $\mathsf{k}, \mathsf{h} < 1$, we have $\lim\limits_{n \to +\infty} \mathsf{k}^{\mathsf{n}} = \lim\limits_{n \to +\infty} \mathsf{h}^{\mathsf{n}} = 0$ in both cases, which shows that $\{\varkappa_n\}$ is a Cauchy sequence in complete dq-ultrametric space (Y, d_u).
Since $\top$ is continuous, there is a point $\varkappa \in Y$ such that

$$\lim_{n \to +\infty} \varkappa_n = \varkappa.$$

Then

$$\begin{aligned}\top\varkappa &= \top \lim_{n\to+\infty} \varkappa_n \\ &= \lim_{n\to+\infty} \top\varkappa_n \\ &= \lim_{n\to+\infty} \varkappa_{n+1} \\ &= \varkappa.\end{aligned}$$

Uniqueness part: Suppose that $\varkappa$ and γ are two different fixed points of $\top$. Then

$$\top\varkappa = \varkappa \text{ and } \top\gamma = \gamma.$$

We assert that $d_u(\varkappa,\varkappa) = d_u(\gamma,\gamma) = 0$. If $d_u(\varkappa,\varkappa) > 0$ and $d_u(\gamma,\gamma) > 0$, then we derive from (1) that

$$\begin{aligned}d_u(\varkappa,\varkappa) &= d_u(\top\varkappa,\top\varkappa) \\ &\leq \delta d_u(\varkappa,\varkappa) + \beta d_u(\varkappa,\top\varkappa) + \gamma d_u(\varkappa,\top\varkappa)\end{aligned}$$

$$d_u(\varkappa,\varkappa) \leq (\delta+\beta+\gamma) d_u(\varkappa,\varkappa)$$

and

$$d_u(\gamma,\gamma) \leq (\delta+\beta+\gamma) d_u(\gamma,\gamma)$$

respectively, which is a contradiction to $0 < \delta+\beta+2\gamma < 1$.

Assume now that $d_u(\varkappa,\gamma) > 0$ and $d_u(\gamma,\varkappa) > 0$. Then we get the following

$$d_u(\varkappa,\gamma) \leq (\delta+\gamma) d_u(\varkappa,\gamma) + \beta d_u(\gamma,\varkappa) \tag{2}$$

and similarly

$$d_u(\gamma,\varkappa) \leq \beta d_u(\varkappa,\gamma) + (\delta+\gamma) d_u(\gamma,\varkappa). \tag{3}$$

Combining (2) and (3), we get

$$|d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)| \leq |(\delta-\beta+\gamma)||d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)|$$

which implies that, $d_u(\varkappa,\gamma) = d_u(\gamma,\varkappa)$. Since $0 \leq \delta-\beta+\gamma < 1$, it follows from (1) that

$$\begin{aligned}d_u(\varkappa,\gamma) &\leq \delta d_u(\varkappa,\gamma) + \beta d_u(\gamma,\top\varkappa) + \gamma d_u(\varkappa,\top\gamma) \\ &\leq \delta d_u(\varkappa,\gamma) + \beta d_u(\gamma,\varkappa) + \gamma d_u(\varkappa,\gamma) \\ &\leq (\delta+\beta+\gamma) d_u(\varkappa,\gamma),\end{aligned}$$

which gives $d_u(\varkappa,\gamma) = 0$, since $0 \leq (\delta+\beta+\gamma) < 1$. Further $d_u(\varkappa,\gamma) = d_u(\gamma,\varkappa) = 0$. This fact yields $\varkappa = \gamma$. Hence the proof is complete. □

Example 2.3. Let (Y, d_u) be a complete dislocated quasi-ultrametric space with $Y = [0,1]$, where $d_u(\varkappa,\gamma) = |\varkappa-\gamma|_p + 3|\varkappa|_p + 2|\gamma|_p,\ for\ all\ \varkappa,\gamma \in Y$. Define the continuous self-mapping $\top$ on Y by $\top\varkappa = \frac{-\varkappa}{7}$ satisfying Theorem 2.1. Then $\top$ has a unique fixed point.

Now, we consider

$$\begin{aligned} d_u(\top\varkappa, \top\gamma) &= |\top\varkappa - \top\gamma|_p + 3|\frac{-\varkappa}{7}|_p + 2|\frac{-\gamma}{7}|_p \\ &\le \delta(|\varkappa-\gamma|_p + 3|\varkappa|_p + 2|\gamma|_p) + \beta(|\gamma + \frac{\varkappa}{7}|_p + 3|\gamma|_p + 2|\frac{\varkappa}{7}|_p) \\ &\qquad + \gamma(|\varkappa + \frac{\gamma}{7}|_p + 3|\varkappa|_p + 2|\frac{\gamma}{7}|_p) \\ &\le \delta(|\varkappa-\gamma|_p + 3|\varkappa|_p + 2|\gamma|_p) + \beta(|\gamma - \top\varkappa|_p + 3|\gamma|_p + 2|\top\varkappa|_p) \\ &\qquad + \gamma(|\varkappa - \top\gamma|_p + 3|\varkappa|_p + 2|\top\gamma|_p) \\ &\le \delta d_u(\varkappa,\gamma) + \beta d_u(\gamma, \top\varkappa) + \gamma d_u(\varkappa, \top\gamma). \end{aligned}$$

Then the contraction condition in Theorem 2.1 holds by selecting proper values of δ, β, γ in $[0, 1)$ and p such that $0 \le \delta + \beta + \gamma < 1$. Therefore $\top$ has a unique fixed point.

Theorem 2.2. *Let (Y, d_u) be a complete* dq*-ultrametric space and $\top : Y \to Y$ be a continuous self mapping satisfying the following contraction*

$$d_u(\top\varkappa, \top\gamma) \le \delta d_u(\varkappa,\gamma) + \beta d_u(\varkappa, \top\varkappa) + \gamma d_u(\gamma, \top\gamma) + e[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)] \tag{4}$$

where $\delta, \beta, \gamma, e \in [0, 1)$ with $\delta + \beta + \gamma + 2e < 1$. Then $\top$ has a unique fixed point.

Proof. Let $\varkappa_0$ be chosen arbitrarily. Then we define a sequence $\{\varkappa_n\}$ by the rule $\varkappa_0, \varkappa_1 = \top\varkappa_0,\ \ \varkappa_2 = \top\varkappa_1, \cdots \varkappa_{n+1} = \top\varkappa_n,\quad$ for all $\mathfrak{n} \in \mathbb{N}$.

Now we show that $\varkappa_n$ is a Cauchy sequence in Y. Suppose that

$$d_u(\varkappa_n, \varkappa_{n+1}) = d_u(\top\varkappa_{n-1}, \top\varkappa_n).$$

Then

$$\begin{aligned} &d_u(\varkappa_n, \varkappa_{n+1}) \\ &\le \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \top\varkappa_{n-1}) + \gamma d_u(\varkappa_n, \top\varkappa_n) + e[d_u(\varkappa_{n-1}, \top\varkappa_n) + d_u(\varkappa_n, \top\varkappa_{n-1})] \\ &\le \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma d_u(\varkappa_n, \varkappa_{n+1}) + e[d_u(\varkappa_{n-1}, \varkappa_{n+1}) + d_u(\varkappa_n, \varkappa_n)] \\ &\le \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma d_u(\varkappa_n, \varkappa_{n+1}) \\ &+ e \max\left\{ d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1}) \right\} + e[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]. \end{aligned}$$

Case 3: If $\max\left\{ d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1}) \right\} = d_u(\varkappa_{n-1}, \varkappa_n)$, then we get

$$\begin{aligned} d_u(\varkappa_n, \varkappa_{n+1}) &\le \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma d_u(\varkappa_n, \varkappa_{n+1}) + 2e d_u(\varkappa_{n-1}, \varkappa_n) + e d_u(\varkappa_n, \varkappa_{n+1}) \\ &\le \left(\frac{\delta + \beta + 2e}{1 - \gamma - e} \right) d_u(\varkappa_{n-1}, \varkappa_n) \\ &\le \mathsf{k}_1 \ \ d_u(\varkappa_{n-1}, \varkappa_n), \end{aligned}$$

where $\mathsf{k}_1 = \frac{\delta+\beta+2e}{1-\gamma-e} < 1$, and so

$$d_u(\varkappa_n, \varkappa_{n+1}) \le \mathsf{k}_1^n \ \ d_u(\varkappa_0, \varkappa_1).$$

Case 4: If $\max\left\{d_u(\varkappa_{n-1},\varkappa_n), d_u(\varkappa_n,\varkappa_{n+1})\right\} = d_u(\varkappa_n,\varkappa_{n+1})$, then we get

$$\begin{aligned} d_u(\varkappa_n,\varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1},\varkappa_n) + \beta d_u(\varkappa_{n-1},\varkappa_n) + \gamma d_u(\varkappa_n,\varkappa_{n+1}) + e d_u(\varkappa_{n-1},\varkappa_n) + 2e d_u(\varkappa_n,\varkappa_{n+1}) \\ &\leq \left(\frac{\delta+\beta+e}{1-\gamma-2e}\right) d_u(\varkappa_{n-1},\varkappa_n) \\ &\leq \mathsf{h}_1\ d_u(\varkappa_{n-1},\varkappa_n), \end{aligned}$$

where $\mathsf{h}_1 = \frac{\delta+\beta+e}{1-\gamma-2e} < 1$, and so

$$d_u(\varkappa_n,\varkappa_{n+1}) \leq \mathsf{h}_1^n\ d_u(\varkappa_0,\varkappa_1).$$

Since $\mathsf{k}_1, \mathsf{h}_1 < 1$, we have $\lim\limits_{n\to+\infty} \mathsf{k}_1^n = \lim\limits_{n\to+\infty} \mathsf{h}_1^n = 0$ in both cases, which shows that $\{\varkappa_n\}$ is a Cauchy sequence in the complete dq-ultrametric space (Y, d_u).

By virtue of the fact that $\top$ is continuous, there exists a point $\varkappa \in Y$ such that

$$\lim_{n\to+\infty} \varkappa_n = \varkappa.$$

Then

$$\begin{aligned} \top\varkappa &= \top \lim_{n\to+\infty} \varkappa_n \\ &= \lim_{n\to+\infty} \top\varkappa_n \\ &= \lim_{n\to+\infty} \varkappa_{n+1} \\ &= \varkappa. \end{aligned}$$

Uniqueness part: Suppose that $\varkappa$ and γ are two different fixed points of $\top$. Then

$$\top\varkappa = \varkappa \text{ and } \top\gamma = \gamma.$$

We assert that $d_u(\varkappa,\varkappa) = d_u(\gamma,\gamma) = 0$. If $d_u(\varkappa,\varkappa) > 0$ and $d_u(\gamma,\gamma) > 0$, then we derive from (4) that

$$\begin{aligned} d_u(\varkappa,\varkappa) &= d_u(\top\varkappa,\top\varkappa) \\ &\leq \delta d_u(\varkappa,\varkappa) + \beta d_u(\varkappa,\top\varkappa) + \gamma d_u(\varkappa,\top\varkappa) + e[d_u(\varkappa,\top\varkappa) + d_u(\varkappa,\top\varkappa)] \\ d_u(\varkappa,\varkappa) &\leq (\delta+\beta+\gamma+2e) d_u(\varkappa,\varkappa) \end{aligned}$$

and similarly

$$d_u(\gamma,\gamma) \leq (\delta+\beta+\gamma+2e) d_u(\gamma,\gamma)$$

respectively, which is a contradiction to $0 < \delta+\beta+\beta+2e < 1$. Assume now that $d_u(\varkappa,\gamma) > 0$ and $d_u(\gamma,\varkappa) > 0$. Then we get the following

$$d_u(\varkappa,\gamma) \leq (\delta+e) d_u(\varkappa,\gamma) + e d_u(\gamma,\varkappa).$$

Similarly,

$$d_u(\gamma,\varkappa) \leq e d_u(\varkappa,\gamma) + (\delta+e) d_u(\gamma,\varkappa),$$

$$|d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)| \leq |\delta||d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)|,$$

which implies that, $d_u(\varkappa,\gamma) = d_u(\gamma,\varkappa)$, since $0 \leq \delta < 1$. It follows from from (4) that

$$d_u(\varkappa,\gamma) \leq (\delta+2e) d_u(\varkappa,\gamma),$$

which gives $d_u(\varkappa,\gamma) = 0$, since $0 \leq \delta+2e < 1$. Further $d_u(\varkappa,\gamma) = d_u(\gamma,\varkappa) = 0$. This fact yields $\varkappa = \gamma$. Hence the proof is complete. □

Theorem 2.3. *Let (Y, d_u) be a complete* dq*-ultrametric space and $\top : Y \to Y$ be a continuous self map satisfying the following contraction*

$$d_u(\top\varkappa, \top\gamma) \leq \delta d_u(\varkappa, \gamma) + \beta d_u(\varkappa, \top\varkappa) + \gamma d_u(\gamma, \top\varkappa) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)], \quad (5)$$

where $\delta, \beta, \gamma, e, f \in [0, 1)$ with $\delta + \beta + \gamma + 2e + 2f < 1$. Then $\top$ has a unique fixed point.

Proof. Let $\varkappa_0$ be chosen arbitrarily. Then we define a sequence $\{\varkappa_n\}$ by the rule $\varkappa_0, \varkappa_1 = \top\varkappa_0,\ \varkappa_2 = \top\varkappa_1, \cdots \varkappa_{n+1} = \top\varkappa_n,$ for all $\mathfrak{n} \in \mathbb{N}$.

Now we show that $\{\varkappa_n\}$ is a Cauchy sequence in Y. Suppose that

$$d_u(\varkappa_n, \varkappa_{n+1}) = d_u(\top\varkappa_{n-1}, \top\varkappa_n).$$

Then

$$\begin{aligned}
&d_u(\varkappa_n, \varkappa_{n+1})\\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \top\varkappa_{n-1}) + \beta d_u(\varkappa_n, \top\varkappa_{n-1})\\
&+ e[d_u(\varkappa_{n-1}, \top\varkappa_{n-1}) + d_u(\varkappa_n, \top\varkappa_n)] + f[d_u(\varkappa_{n-1}, \top\varkappa_n) + d_u(\varkappa_n, \top\varkappa_{n-1})]\\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]\\
&+ e[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})] + f[d_u(\varkappa_{n-1}, \varkappa_{n+1}) + d_u(\varkappa_n, \varkappa_n)]\\
&\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]\\
&+ e[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]\\
&+ f[\max\left\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\right\} + d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})].
\end{aligned}$$

Case 5: If $\max\left\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\right\} = d_u(\varkappa_{n-1}, \varkappa_n)$, then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]\\
&+ e[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})] + 2f d_u(\varkappa_{n-1}, \varkappa_n) + e d_u(\varkappa_n, \varkappa_{n+1})\\
&\leq \left(\frac{\delta + \beta + \gamma + e + 2f}{1 - \gamma - e - f}\right) d_u(\varkappa_{n-1}, \varkappa_n)\\
&\leq \mathsf{k}_2\ d_u(\varkappa_{n-1}, \varkappa_n),
\end{aligned}$$

where $\mathsf{k}_2 = \frac{\delta+\beta+\gamma+e+2f}{1-\gamma-e-f} < 1$, and so

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{k}_2^n\ d_u(\varkappa_0, \varkappa_1) \text{ for all } n \in \mathbb{N}.$$

Case 6: If $\max\left\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\right\} = d_u(\varkappa_n, \varkappa_{n+1})$, then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \beta d_u(\varkappa_{n-1}, \varkappa_n) + \gamma[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})]\\
&+ e[d_u(\varkappa_{n-1}, \varkappa_n) + d_u(\varkappa_n, \varkappa_{n+1})] + f d_u(\varkappa_{n-1}, \varkappa_n) + 2f d_u(\varkappa_n, \varkappa_{n+1})\\
&\leq \left(\frac{\delta + \beta + \gamma}{+} e + f 1 - \gamma - e - 2f\right) d_u(\varkappa_{n-1}, \varkappa_n)\\
&\leq \mathsf{h}_2\ d_u(\varkappa_{n-1}, \varkappa_n),
\end{aligned}$$

where $\mathsf{h}_2 = \frac{\delta+\beta+\gamma+e+f}{1-\gamma-e-2f} < 1$, and so

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{h}_2^n\ d_u(\varkappa_0, \varkappa_1) \text{ for all } n \in \mathbb{N}.$$

Since $\mathsf{k}_2, \mathsf{h}_2 < 1$, we have $\lim_{n\to+\infty} \mathsf{k}_2^n = \lim_{n\to+\infty} \mathsf{h}_2^n = 0$ in both cases, which shows that $\{\varkappa_n\}$ is a Cauchy.

By virtue of the fact that $\top$ is continuous, there exists a point $\varkappa \in Y$ such that

$$\lim_{n\to+\infty} \varkappa_n = \varkappa.$$

Then

$$\begin{aligned} \top\varkappa &= \top \lim_{n\to+\infty} \varkappa_n \\ &= \lim_{n\to+\infty} \top\varkappa_n \\ &= \lim_{n\to+\infty} \varkappa_{n+1} \\ &= \varkappa. \end{aligned}$$

Uniqueness part: Suppose that $\varkappa$ and γ are two different fixed points of $\top$. Then

$$\top\varkappa = \varkappa \text{ and } \top\gamma = \gamma.$$

We assert that $d_u(\varkappa,\varkappa) = d_u(\gamma,\gamma) = 0$. If $d_u(\varkappa,\varkappa) > 0$ and $d_u(\gamma,\gamma) > 0$, then we derive from (5) that

$$\begin{aligned} d_u(\varkappa,\varkappa) &= d_u(\top\varkappa,\top\varkappa) \\ &\le \delta d_u(\varkappa,\varkappa) + \delta d_u(\varkappa,\top\varkappa) + \gamma d_u(\varkappa,\top\varkappa) + e[d_u(\varkappa,\top\varkappa) + d_u(\varkappa,\top\varkappa)] \\ &\quad + f[d_u(\varkappa,\top\varkappa) + d_u(\varkappa,\top\varkappa)] \\ &\le (\delta+\beta+\gamma+2e+2f) d_u(\varkappa,\varkappa) \end{aligned}$$

and similarly

$$d_u(\gamma,\gamma) \le (\delta+\beta+\gamma+2e+2f) d_u(\gamma,\gamma),$$

respectively, which is a contradiction to $0 < \delta+\beta+\gamma+2e+2f < 1$. Assume now that $d_u(\varkappa,\gamma) > 0$ and $d_u(\gamma,\varkappa) > 0$. Then we get the following

$$d_u(\varkappa,\gamma) \le (\delta+f) d_u(\varkappa,\gamma) + (\gamma+f) d_u(\gamma,\varkappa)$$

and similarly

$$d_u(\gamma,\varkappa) \le (\gamma+f) d_u(\varkappa,\gamma) + (\beta+f) d_u(\gamma,\varkappa).$$

Thus

$$|d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)| \le |\delta-\gamma||d_u(\varkappa,\gamma) - d_u(\gamma,\varkappa)|,$$

which gives $d_u(\varkappa,\gamma) = 0$, since $0 \le \delta + 2e < 1$. Further $d_u(\varkappa,\gamma) = d_u(\gamma,\varkappa) = 0$. This fact yields $\varkappa = \gamma$. The proof is complete. □

Example 2.4. Let $Y = [0,1]$ with a complete dislocated quasi ultrametric space defined by $d_u(\varkappa,\gamma) = |\varkappa|_p$ *for all* $\varkappa,\gamma \in Y$, and define the continuous self mapping $\top$ defined by $\top\varkappa = 8\varkappa$ satisfying Theorem 2.3. Then $\top$ has a unique fixed point.

Proof. Assume that δ,β,γ,e,f lie between 0 and 1, with $\delta+\beta+\gamma+2e+2f < 1$. Suppose that $\delta = \frac{1}{5}, \beta = \frac{1}{6}, \gamma = \frac{1}{8}$ and $e = \frac{1}{10}, f = \frac{1}{12}$. Let $\varkappa$ and γ be fixed such that $\varkappa = \frac{1}{2}$ and $\gamma = \frac{1}{3}$. Using the inequality (5) we obtain the following results

(1) $d_u(\top\varkappa,\top\gamma) = |4|_p$,
(2) $d_u(\varkappa,\gamma) = d_u(\varkappa,\top\gamma) = d_u(\varkappa,\top\varkappa) = |\frac{1}{2}|_p$,
(3) $d_u(\gamma,\top\varkappa) = d_u(\gamma,\top\gamma) = |\frac{1}{3}|_p$.

When $p = 2$,

$$\frac{1}{4} \leq 2\delta + 2\beta + \gamma + 3e + 3f$$

and hence clearly $\top$ satisfies the inequality (5) and $\varkappa = 0$ is the unique fixed point of $\top \in Y$. $\square$

Corollary 2.1. *Let (Y, d_u) be a complete* dq*-ultrametric space and $\top : Y \to Y$ be a continuous self mapping satisfying the following contraction conditions: For all $\delta, \beta, \gamma, e, f \in Y$.*

(1) $d_u(\top\varkappa, \top\gamma)$
$\leq \delta d_u(\varkappa, \gamma) + \gamma d_u(\gamma, \top\varkappa) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\delta, \gamma, e, f \in [0, 1)$ with $\delta + \gamma + 2e + 2f < 1$;
(2) $d_u(\top\varkappa, \top\gamma) \leq \delta d_u(\varkappa, \gamma) + \gamma d_u(\gamma, \top\varkappa) + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\delta, \gamma, f \in [0, 1)$ with $\delta + \gamma + 2f < 1$;
(3) $d_u(\top\varkappa, \top\gamma) \leq \delta d_u(\varkappa, \gamma) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\delta, e, f \in [0, 1)$ with $\delta + 2e + 2f < 1$;
(4) $d_u(\top\varkappa, \top\gamma) \leq \gamma d_u(\gamma, \top\varkappa) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\gamma, e, f \in [0, 1)$ with $\gamma + 2e + 2f < 1$.

Then $\top$ has a unique fixed point.

Corollary 2.2. *Let (Y, d_u) be a complete* dq*-ultrametric space and $\top : Y \to Y$ be a continuous self mapping satisfying the following contraction conditions: For all $\delta, \beta, \gamma, e, f \in Y$.*

(1) $d_u(\top\varkappa, \top\gamma)$
$\leq \beta d_u(\varkappa, \top\varkappa) + \gamma d_u(\gamma, \top\varkappa) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\beta, \gamma, e, f \in [0, 1)$ with $\beta + \gamma + 2e + 2f < 1$;
(2) $d_u(\top\varkappa, \top\gamma) \leq \beta d_u(\varkappa, \top\varkappa) + \gamma d_u(\gamma, \top\varkappa) + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where
$beta, \gamma, f \in [0, 1)$ with $\beta + \gamma + 2f < 1$;
(3) $d_u(\top\varkappa, \top\gamma) \leq \beta d_u(\varkappa, \top\varkappa) + e[d_u(\varkappa, \top\varkappa) + d_u(\gamma, \top\gamma)] + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\beta, e, f \in [0, 1)$ with $\beta + 2e + 2f < 1$;
(4) $d_u(\top\varkappa, \top\gamma) \leq \delta d_u(\varkappa, \gamma) + \beta d_u(\varkappa, \top\varkappa) + f[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)]$,
where $\delta, \beta, f \in [0, 1)$ with $\delta + \beta + 2f < 1$.

Then $\top$ has a unique fixed point.

2.2. **Fixed-point results are obtained in dislocated quasi-ultrametric spaces using rational contractions.**

Theorem 2.4. *Let (Y, d_u) be a complete* dq*-ultrametric space, and let $\top : Y \to Y$ be a continuous self mapping such that for all $\varkappa, \gamma \in Y$, $0 \leq \delta, \beta < 1$, and $3\delta + \beta < 1$,*

$$d_u(\top\varkappa, \top\gamma) \leq \delta[d_u(\varkappa, \top\gamma) + d_u(\gamma, \top\varkappa)] + \beta \frac{d_u(\gamma, \top\gamma)[1 + d_u(\varkappa, \top\varkappa)]}{1 + d_u(\varkappa, \gamma)}.$$

Then there exists $\varkappa \in Y$ such that $\top\varkappa = \varkappa$.

Proof. Suppose $\varkappa_0 \in Y$ and $\{\varkappa_n\}$ is a sequence in Y such that $\top\varkappa_n = \varkappa_{n+1}$. Then

$$\begin{aligned}
&d_u(\varkappa_n, \varkappa_{n+1})\\
&= d_u(\top\varkappa_{n-1}, \top\varkappa_n)\\
&\leq \delta[d_u(\varkappa_{n-1}, \top\varkappa_n) + d_u(\varkappa_n, \top\varkappa_{n-1})] + \beta\frac{d_u(\varkappa_n, \top\varkappa_n)[1 + d_u(\varkappa_{n-1}, \top\varkappa_{n-1})]}{1 + d_u(\varkappa_{n-1}, \varkappa_n)}\\
&\leq \delta[d_u(\varkappa_{n-1}, \varkappa_{n+1}) + d_u(\varkappa_n, \varkappa_n)] + \beta\frac{d_u(\varkappa_n, \varkappa_{n+1})[1 + d_u(\varkappa_{n-1}, \varkappa_n)]}{1 + d_u(\varkappa_{n-1}, \varkappa_n)}\\
&\leq \delta[d_u(\varkappa_{n-1}, \varkappa_{n+1}) + d_u(\varkappa_n, \varkappa_n)] + \beta d_u(\varkappa_n, \varkappa_{n+1})\\
&\leq \delta \max[d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})] + \delta[d_u(\varkappa_n, \varkappa_{n-1}) + d_u(\varkappa_{n-1}, \varkappa_n)] + \beta d_u(\varkappa_n, \varkappa_{n+1}).
\end{aligned}$$

Case 7: If $\max\left\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\right\} = d_u(\varkappa_{n-1}, \varkappa_n)$, then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n) + \delta[d_u(\varkappa_n, \varkappa_{n-1}) + d_u(\varkappa_{n-1}, \varkappa_n)] + \beta d_u(\varkappa_n, \varkappa_{n+1}),\\
(1 - \delta - \beta)d_u(\varkappa_n, \varkappa_{n+1}) &\leq 2\delta d_u(\varkappa_{n-1}, \varkappa_n)\\
&\leq \left(\frac{2\delta}{1 - (\delta + \beta)}\right)d_u(\varkappa_{n-1}, \varkappa_n)
\end{aligned}$$

where $\mathsf{k}_3 = \frac{2\delta}{1-(\delta\beta)} < 1$, and so

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{k}_3^n \ d_u(\varkappa_0, \varkappa_1) \text{ for all } n \in \mathbb{N}.$$

Case 8: If $\max\left\{d_u(\varkappa_{n-1}, \varkappa_n), d_u(\varkappa_n, \varkappa_{n+1})\right\} = d_u(\varkappa_n, \varkappa_{n+1})$, then we get

$$\begin{aligned}
d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_n, \varkappa_{n+1}) + \delta[d_u(\varkappa_n, \varkappa_{n-1}) + d_u(\varkappa_{n-1}, \varkappa_n)] + \beta d_u(\varkappa_n, \varkappa_{n+1}),\\
(1 - 2\delta - \beta)d_u(\varkappa_n, \varkappa_{n+1}) &\leq \delta d_u(\varkappa_{n-1}, \varkappa_n)\\
&\leq \left(\frac{\delta}{1 - (2\delta + \beta)}\right)d_u(\varkappa_{n-1}, \varkappa_n)
\end{aligned}$$

where $\mathsf{h}_3 = \frac{\delta}{1-(2\delta+\beta)} < 1$, and so

$$d_u(\varkappa_n, \varkappa_{n+1}) \leq \mathsf{h}_3^n \ d_u(\varkappa_0, \varkappa_1) \text{ for all } n \in \mathbb{N}.$$

Since $\mathsf{k}_3, \mathsf{h}_3 < 1$, we have $\lim\limits_{n\to+\infty} \mathsf{k}_3^n = \lim\limits_{n\to+\infty} \mathsf{h}_3^n = 0$ in both cases, which shows that $\{\varkappa_n\}$ is a Cauchy sequence in the complete dq-ultrametric space (Y, d_u).

By virtue of the fact that $\top$ is continuous, there exists a point $\varkappa \in Y$ such that

$$\lim_{n\to+\infty} \varkappa_n = \varkappa.$$

Then

$$\begin{aligned}
\top\varkappa &= \top \lim_{n\to+\infty} \varkappa_n\\
&= \lim_{n\to+\infty} \top\varkappa_n\\
&= \lim_{n\to+\infty} \varkappa_{n+1}\\
&= \varkappa.
\end{aligned}$$

Uniqueness part: Suppose that $\varkappa$ and γ are two different fixed points of $\top$. Then

$$\top\varkappa = \varkappa \text{ and } \top\gamma = \gamma.$$

We assert that $d_u(\varkappa,\varkappa)=d_u(\gamma,\gamma)=0$. If $d_u(\varkappa,\varkappa)>0$ and $d_u(\gamma,\gamma)>0$, then we derive from (5) that

$$\begin{aligned} d_u(\varkappa,\varkappa) &= d_u(\top\varkappa,\top\varkappa)\\ &\le \delta[d_u(\varkappa,\top\varkappa)+d_u(\varkappa,\top\varkappa)]+\beta\frac{d_u(\varkappa,\top\varkappa)[1+d_u(\varkappa,\top\varkappa)]}{1+d_u(\varkappa,\varkappa)}\\ &\le (2\delta+\beta)d_u(\varkappa,\varkappa)\end{aligned}$$

and similarly

$$d_u(\gamma,\gamma)\le(2\delta+\beta)d_u(\gamma,\gamma),$$

respectively, which is a contradiction to $2\beta+\beta<1$. Hence $d_u(\varkappa,\varkappa)=0$, similarly we get $d_u(\gamma,\gamma)=0$. Assume that $d_u(\varkappa,\gamma)>0$ and $d_u(\gamma,\varkappa)>0$. Then we get the following

$$\begin{aligned} d_u(\varkappa,\gamma) &= d_u(\top\varkappa,\top\gamma)\\ &\le \delta[d_u(\varkappa,\top\gamma)+d_u(\gamma,\top\varkappa)]+\beta\frac{d_u(\gamma,\top\gamma)[1+d_u(\varkappa,\top\varkappa)]}{1+d_u(\varkappa,\gamma)}\\ &\le \delta[d_u(\varkappa,\gamma)+d_u(\gamma,\varkappa)]+\beta\frac{d_u(\gamma,\gamma)[1+d_u(\varkappa,\varkappa)]}{1+d_u(\varkappa,\gamma)}\end{aligned}$$

and similarly

$$\begin{aligned} d_u(\gamma,\varkappa) &= d_u(\top\gamma,\top\varkappa)\\ &\le \delta[d_u(\gamma,\top\varkappa)+d_u(\varkappa,\top\gamma)]+\beta\frac{d_u(\varkappa,\top\varkappa)[1+d_u(\gamma,\top\gamma)]}{1+d_u(\gamma,\varkappa)}\\ &\le \delta[d_u(\gamma,\varkappa)+d_u(\varkappa,\gamma)]+\beta\frac{d_u(\varkappa,\varkappa)[1+d_u(\gamma,\gamma)]}{1+d_u(\gamma,\varkappa)}.\end{aligned}$$

Thus

$$|d_u(\varkappa,\gamma)-d_u(\gamma,\varkappa)|=0 \implies d_u(\varkappa,\gamma)=d_u(\gamma,\varkappa). \tag{6}$$

So we get $d_u(\varkappa,\gamma)=d_u(\gamma,\varkappa)=0$. This fact yields $\varkappa=\gamma$. □

Theorem 2.5. *Let $(\mathbb{Y},d_u)$ be a complete dq-ultrametric space and $\top:\mathbb{Y}\to\mathbb{Y}$ be a continuous self mapping such that for all $\varkappa,\gamma\in\mathbb{Y}$, $0\le\delta,\beta<1$, and $\delta+3\beta+\gamma<1$,*

$$\begin{aligned}& d_u(\top\varkappa,\top\gamma)\\ &\le \delta[d_u(\varkappa,\top\varkappa)+d_u(\gamma,\top\gamma)]+\beta[d_u(\varkappa,\top\gamma)+d_u(\gamma,\top\varkappa)]+\gamma\frac{d_u(\gamma,\top\gamma)[1+d_u(\varkappa,\top\varkappa)]}{1+d_u(\varkappa,\gamma)}.\end{aligned}$$

Then there exists $\varkappa\in\mathbb{Y}$ such that $\top\varkappa=\varkappa$.

3. Applications

3.1. Applications to integral equations.

The application of fixed point theory in dislocated quasi-ultrametric spaces provides a robust framework for solving integral equations. Integral equations, which are equations where an unknown function appears under an integral sign, arise in various mathematical and applied science fields, such as physics, engineering, and biology. By leveraging the properties of dislocated quasi-ultrametric spaces, one can establish conditions under which solutions to these equations exist and are unique. A dislocated quasi-ultrametric space generalizes the concept of metric spaces by relaxing certain axioms. Specifically, the triangle inequality is replaced with a stronger ultrametric condition, and the self-distance of a point need not be zero. This flexibility makes dislocated quasi-ultrametric spaces particularly suitable for studying problems with non-standard or irregular structures, such as those encountered in integral equations.

Theorem 3.1. *Let $(\mathbb{Y}, d_u)$ be a complete* dq *ultrametric space and $\top : \mathbb{Y} \to \mathbb{Y}$ be a continuous self-mapping. Assume that $\top$ satisfies the following contraction condition:*

$$\begin{aligned} d_u(\top\varkappa, \top\gamma) &\leq \delta\, d_u(\varkappa,\gamma) + \beta\, d_u(\varkappa, \top\varkappa) + \gamma\, d_u(\gamma, \top\varkappa) \\ &\quad + e\big[d_u(\varkappa,\top\varkappa) + d_u(\gamma,\top\gamma)\big] + f\big[d_u(\varkappa,\top\gamma) + d_u(\gamma,\top\varkappa)\big], \end{aligned}$$

for all $\varkappa, \gamma \in \mathbb{Y}$, where the constants $\delta, \beta, \gamma, e, f \in [0,1)$ satisfy the inequality

$$\delta + \beta + \gamma + 2e + 2f < 1.$$

Then $\top$ has a unique fixed point in $(\mathbb{Y}, d_u)$.

Let $W = C([0,1], \mathbb{R}^+)$ be the family of continuous functions defined on $[0,1]$. Consider the following integral equation

$$\varkappa(e) = \int_0^e H(e, f, \varkappa(f))df \tag{7}$$

for all $e \in [0,1]$, where $H : [0,1] \times W \to \mathbb{R}$. For $\varkappa \in C([0,1], \mathbb{R}^+)$, define supremum norm as $\|\varkappa\| = \sup_{s\in[0,1]}\{|\varkappa(s)|e^s\}$ and for all $\varkappa, \gamma \in C([0,1], \mathbb{R}^+)$, define

$$\begin{aligned} d_u(\varkappa,\gamma) &= \frac{1}{2} \sup_{s\in[0,1]} \{|\varkappa(s) + \gamma(s)|e^s\} \\ &= \frac{1}{2}\|\varkappa + \gamma\|. \end{aligned}$$

It is clear that $\mathsf{C}([0,1], \mathbb{R}^+, d)$ is a complete dislocated quasi ultrametric space. So we have the following result.

Theorem 3.2. *Suppose that*

(i) $H : [0,1] \times W \to \mathbb{R}$;
(ii) *Define*

$$(\top\varkappa)(e) = \int_0^e H(e, f, \varkappa(f))df,$$

such that

$$|H(e, f, \varkappa(f)) + H(e, f, \gamma(f))| \leq \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma) + 1}$$

for all $e, f \in [0,1]$ and $\varkappa, \gamma \in C([0,1], \mathbb{R}^+)$, where

$$\begin{aligned} M(\varkappa,\gamma) = \Big(&\delta\|\varkappa+\gamma\| + \beta\|\varkappa + \top\varkappa\| + \gamma\|\gamma + \top\varkappa\| + e(\|\varkappa + \top\varkappa\| + \|\gamma + \top\gamma\|) \\ &+ f(\|\varkappa + \top\gamma\| + \|\gamma + \top\varkappa\|)\Big). \end{aligned}$$

Then (7) posses a unique solution.

Proof. By (ii), we have

$$\begin{aligned}
|\top\varkappa + \top\gamma| &= \int_0^e |H(e,f,\varkappa(f)) + H(e,f,\gamma(f))|\, df \\
&\leq \int_0^e \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma)+1} e^f\, df \\
&\leq \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma)+1} \int_0^e e^f\, df \\
&\leq \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma)+1} e^e.
\end{aligned}$$

This implies

$$\begin{aligned}
|\top\varkappa + \top\gamma| &\leq \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma)+1}, \\
\|\top\varkappa + \top\gamma\| &\leq \frac{M(\varkappa,\gamma)}{M(\varkappa,\gamma)+1}, \\
\frac{M(\varkappa,\gamma)+1}{M(\varkappa,\gamma)} &\leq \frac{1}{\|\top\varkappa + \top\gamma\|}, \\
1 + \frac{1}{M(\varkappa,\gamma)} &\leq \frac{1}{\|\top\varkappa + \top\gamma\|}, \\
1 - \frac{1}{\|\top\varkappa(e) + \top\gamma(e)\|} &\leq \frac{-1}{M(\varkappa,\gamma)}.
\end{aligned}$$

All the conditions of Theorem 2.3 hold and $d_u(\varkappa,\gamma) = \frac{1}{2}\|\varkappa,\gamma\|$. Hence the integral equation (7) admits a unique solution. □

3.2. **Potential applications.**

In dislocated quasi-ultrametric spaces, integral equations find their applications in fixed point theory. This includes various aspects and functions relevant to this area of study.

- **Existence and Uniqueness:** Integral equations make it easy to determine whether fixed points exist for mappings defined on dislocated quasi-ultrametric spaces. By formulating the problem as an integral equation, one can often apply fixed-point theorems to establish the existence of solutions under appropriate conditions.
- **Convergence Analysis:** Integral equations let you study how iterative methods used to get close to fixed points behave when they converge. Understanding the convergence properties of such methods is crucial for developing efficient numerical algorithms.
- **Applications in Dynamic Systems:** We can study dynamical systems controlled by mappings defined in dislocated quasi-ultrametric spaces using integral equations. Fixed-point results obtained through integral equations provide insights into such systems' long-term behavior.

Studying integral equations in dislocated quasi-ultrametric spaces using fixed-point theory provides a new approach to analyzing complex mathematical problems. This method has practical applications in various fields, such as mathematical analysis, functional analysis, and mathematical physics.

4. Conclusions

In our paper, we delve into the intriguing realm of dq-ultrametric spaces, a concept that merges two fundamental notions in mathematics: dq-metric spaces and ultrametric spaces. This fusion opens new avenues for exploring fixed-point theory, a central area in mathematical analysis with broad applications across various disciplines. Firstly, we rigorously define dq-ultrametric spaces, establishing the foundational framework for our subsequent discussions. Our new setting is a mix of the rich theory of dq-metric spaces, which are generalized metric spaces by loosening the triangle inequality, and ultrametric spaces, which are known for having a strong triangular inequality.

Our paper makes a big contribution by showing fixed-point theorems in dq-ultrametric spaces using generalized contractions as the main tool. These theorems build on previous work in both dq-metric spaces and ultrametric spaces. They show how well our method works for bringing together and expanding previous research. We also explain different contractive conditions that are specific to dq-ultrametric spaces. Each one gives us a different view of fixed-point properties. Making use of these conditions, we get a wide range of fixed-point results that show how flexible and rich it is to study dq-ultrametric spaces.

In essence, our work not only establishes the theoretical underpinnings of dq-ultrametric spaces but also demonstrates their practical utility in elucidating fixed-point phenomena. By offering novel perspectives and tools for analysis, we contribute to the ongoing advancement of fixed-point theory and its applications in diverse mathematical contexts.

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